Dr. rer.nat. Julian Köllermeier, Rik Verbiest, M.Sc., Maddalena Dona, M.Sc.

Differential Equations In Science And Engineering | 22/23 IIa
Sample Solution Examination | 06.04.2023

Exercise 1.

Let $k(t)$ be the number of kangaroos in Australia at time t (the prey) and $p(t)$ the number of predators at time t. This prey and predator situation can be described by the system

$$\begin{aligned}\frac{dk}{dt} &= \alpha k - \beta k^2 - \gamma k p,\\ \frac{dp}{dt} &= -\sigma p + \lambda k p,\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma, \lambda \in \mathbb{R}_+$ are non-negative constants.

a) What are the physical interpretations of the constants $\alpha, \beta, \gamma, \sigma, \lambda$?

b) What are the two steady states of the ODE system?

(Note: there are actually three steady states)

c) What are the stability properties of the steady states of the system? Classify the steady states.

0.5+1.5+3 points

Solution.

a)
- α is the specific natural reproduction rate of prey
- β is α divided by the carrying capacity
- γ is the hunting rate
- σ is the death rate of predators
- λ is the birth rate of predators

b) **(Note: there are actually three steady states. Full points were given as soon as any two steady states were correctly derived.)**

The condition for steady state is that temporal derivatives vanish.

We get from $\frac{dk}{dt} = 0$ and $\frac{dp}{dt} = 0$, that the following three steady states exist:

(1) $k = 0$ and $p = 0$,

(2) $k = \frac{\alpha}{\beta}$ and $p = 0$.

(3) $k = \frac{\sigma}{\lambda}$ and $p = \frac{\alpha - \beta\sigma/\lambda}{\gamma}$.

c) **(Note: there are actually three steady states, with difficult stability analysis for the second and third steady state. Full points were given as soon as the first steady state was correctly analysed and at least one other was started up to the point where eigenvalues were given.)**

We compute the Jacobian

$$\begin{pmatrix} \alpha - 2\beta k - \gamma p & -\gamma k \\ \lambda p & -\sigma + \lambda k \end{pmatrix}.$$

For stability, we compute the Jacobians at the steady states and compute their eigenvalues, which characterize the stability:

(1) $\begin{pmatrix} \alpha & 0 \\ 0 & -\sigma \end{pmatrix} \Longrightarrow$ EV: $\lambda_1 = \alpha > 0, \lambda_2 = -\sigma < 0$,

(2) $\begin{pmatrix} -\alpha & -\gamma\alpha/\beta \\ 0 & -\sigma + \lambda\alpha/\beta \end{pmatrix} \Longrightarrow$ EV: $\lambda_{1,1} = -\alpha < 0, \lambda_2 = -\sigma + \lambda\alpha/\beta$,

(3) $\begin{pmatrix} -\beta\sigma/\lambda & -\gamma\sigma/\gamma \\ -\frac{\lambda\alpha-\beta\sigma}{\gamma} & 0 \end{pmatrix} \Longrightarrow$ EV: $\lambda_{1,2} = \ldots$.

We conclude the following stability properties for the steady states:

(1) (unstable) saddle point,

(2) not classifiable without additional knowledge about the coefficients,

(3) not classifiable without additional knowledge about the coefficients.

Exercise 2.
We consider the following system (1) of chemical reactions for the four species A, B, C, D:

$$\begin{aligned} A + B &\xrightarrow{k_1} 2B \\ B + C &\xrightarrow{k_2} 2C \\ C &\xrightarrow{k_3} D \end{aligned} \tag{1}$$

a) Derive the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by A, B, C, D.

b) Draw the reaction network.

c) Show that $A - D$ is a conserved quantity.
(Note: $A - D$ is actually not conserved, but $A + B + C + D$ is.)

2.5+1+1.5 points

Solution.

a) We have 4 substances A, B, C, D, so $N = 4$ and 3 reactions, so $M = 3$.
The stoichiometric coefficients are given by the following table:

$\gamma_{i,m}$	1	2	3
A	-1	0	0
B	1	-1	0
C	0	1	-1
D	0	0	1

The reaction rates are given by

$$\begin{aligned} \lambda_1 &= k_1(T) \cdot n_A \cdot n_B \\ \lambda_2 &= k_2(T) \cdot n_B \cdot n_C \\ \lambda_3 &= k_3(T) \cdot n_C \end{aligned}$$

The production rates R_i of the substances are then given by $R_i = \sum_{m=1}^{3} \gamma_{i,m} \lambda_m$, which means

$$\begin{aligned} R_A &= -\lambda_1 \\ R_B &= -\lambda_1 - \lambda_2 \\ R_C &= \lambda_2 - \lambda_3 \\ R_D &= \lambda_3 \end{aligned}$$

This leads to the following ODE system

$$\begin{aligned} \frac{dn_A}{dt} &= -k_1(T) n_A n_B \\ \frac{dn_B}{dt} &= k_1(T) n_A n_B - k_2(T) n_B n_C \\ \frac{dn_C}{dt} &= k_2(T) n_B n_C - k_3(T) n_C \\ \frac{dn_D}{dt} &= k_3(T) n_C. \end{aligned}$$

b) The reaction network is given by
Full points were only given if it is clear where the reaction happens (indicated by the dots in Figure 1). Otherwise the reaction network would not be generalizable for autocatalytic reactions that have product species.

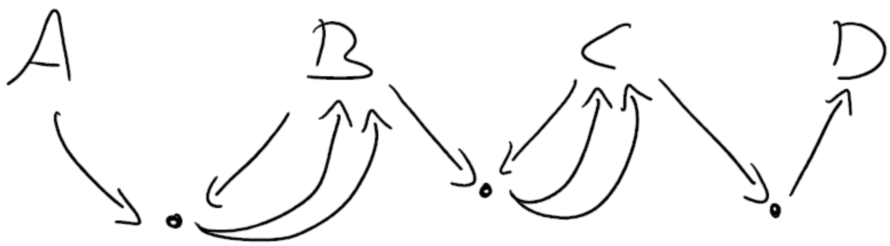

Abbildung 1: Exercise 2 reaction network.

c) **(Note: $A - D$ is actually not conserved, but $A + B + C + D$ is. This means that everybody got the full number of points regardless of the solution here.)**

For conservation, we solve $\sum_{i=1}^{N} \alpha_i \gamma_{i,m} = 0$ for $m = 1, 2, 3$. This leads to

$$
\begin{aligned}
-\alpha_1 + \alpha_2 &= 0 \\
-\alpha_2 + \alpha_3 &= 0 \\
-\alpha_3 + \alpha_4 &= 0.
\end{aligned}
$$

Adding all three equations, we end up with $\alpha_1 = \alpha_2 = alpha_3 = \alpha_4 = 0$. This means that $n_A + n_B + n_C + n_D$ is constant.

Exercise 3.

We consider the scalar, linearized, viscous Burgers equation

$$\frac{\partial}{\partial t}u + u_0^2\frac{\partial}{\partial x}u = D\frac{\partial^2}{\partial x^2}u, \quad u_0 \in \mathbb{R}, D \in \mathbb{R}, \tag{2}$$

with viscosity constant D and advection velocity u_0.
We want to perform a linear stability analysis of equation (2) using the wave ansatz

$$u(t,x) = c \cdot e^{i(kx-\omega t)}, \tag{3}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $c \in \mathbb{R}$.

a) What wave frequencies ω in (3) lead to a stable wave in time?

b) Insert the wave ansatz (3) into the Burgers equation (2) to derive a stability condition for the Burgers equation.

c) Show that the stability condition is equivalent to $D > 0$.
What is the physical interpretation of this stability condition and does it make sense?

1+3+1 points

Solution.

a) Stability in time means that the wave solution does not increase with t.
Using $\omega = Re(\omega) + iIm(\omega))$, we get that

$$i(-\omega t) = -i(Re(\omega) + iIm(\omega))t = -iRe(\omega)t + Im(\omega)t.$$

The term $-iRe(\omega)t$ is an oscillation in time and not increasing with t. The term $Im(\omega)t$ is potentially increasing and has to be smaller than 0 for stability.
This means that $\omega \in \mathbb{C}$ has to have negative imaginary part: $Im(\omega) < 0$.

b) We use that the derivatives of the wave ansatz (3) are

$$\begin{aligned} \frac{\partial}{\partial t}u &= -\omega i u, \\ \frac{\partial}{\partial t}u &= kiu, \\ \frac{\partial^2}{\partial x^2}u &= -k^2 u. \end{aligned}$$

Inserting this into the Burgers equation (2) yields

$$\begin{aligned} \frac{\partial}{\partial t}u + u_0^2\frac{\partial}{\partial x}u &= D\frac{\partial^2}{\partial x^2}u \\ \Rightarrow -\omega i u + u_0^2 kiu &= -Dk^2 u \\ \Rightarrow (u_0^2 ki - \omega i + Dk^2)u &= 0. \end{aligned}$$

Non-trivial solutions are given by

$$u_0^2 ki - \omega i + Dk^2 = 0,$$

which leads to

$$\omega = u_0^2 k - Dk^2 i. \tag{4}$$

Using the solution of part a), stable solutions are given by $Im(\omega) < 0$, such that we get from (4) that

$$Im(\omega) = -Dk^2 < 0.$$

c) From $Im(\omega) = -Dk^2 < 0$, we get that linear stability requires $D > 0$. So the diffusion/viscosity coefficient must be positive (no negative friction). This is consistent with the second law of thermodynamics: the entropy must increase.

Exercise 4.

The shallow water equations for water height $h(t, x)$ and vertical velocity $u(t, x)$ are

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix} = -\frac{1}{\lambda} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{5}$$

where $h(t, x)$ and $u(t, x)$ are the unknowns and g, λ are parameters.

a) What physical interpretations do the equations (5) have and what are the main assumptions for their derivation?

b) Show that the system (5) can be written in the following (so-called primitive variable) form:

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ g & u \end{pmatrix} \cdot \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = -\frac{1}{\lambda h} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{6}$$

c) Show that the propagation speeds of system (5) are given by

$$\sigma(A) = \{u - \sqrt{g \cdot h}, u + \sqrt{g \cdot h}\}.$$

d) How do the propagation speeds of system (5) relate to the dimensionless Froude number $Fr = \frac{u}{\sqrt{gh}}$?

What are different regimes for the flow velocity u in terms of the Froude number?

1.5+1.5+1+1 points

Solution.

a) The first equation is derived from the vertical average of the conservation of mass (also-called continuity equation). The second equation is derived from the vertical average of the conservation of momentum in x-direction.

The main assumptions for the derivation are:

- incompressibility or $\rho = const$.
- shallowness or $\frac{H}{L} = \epsilon \ll 1$.
- flat plane, inclination angle $\phi = 0$.
- friction is modeled using a slip law at the bottom with slip length $\lambda > 0$ (that leads to a relaxation of u towards zero with relaxation time $\frac{1}{\lambda}$).
- gravity is modeled using a gravity constant gravitational acceleration g.

b) The first equation of (6) is derived from the first equation of (5).

The second equation of (6) is derived from the second equation of (5) using the first equation obtained before.

c) The eigenvalues of the system matrix are indeed given by

$$\sigma(A) = \{u - \sqrt{g \cdot h}, u + \sqrt{g \cdot h}\}.$$

d) With the definition of the Froude number $Fr = \frac{u_0}{\sqrt{gh}}$, the eigenvalues are also given by

$$\sigma(A) = \{u \left(1 - \frac{1}{Fr}\right), u \left(1 - \frac{1}{Fr}\right)\}.$$

We can now identify the following regimes for the different Froude numbers:

(1) $Fr < 1$: both eigenvalues have different sign, information transport in both directions, subcritical flow.

(2) $Fr = 1$: the eigenvalues are $0, 2u$. This is the threshold between subcritical and supercritical flow.

(3) $Fr < 1$: both eigenvalues have the same sign, information transport in only in one direction, supercritical flow.